

# Kink in superconducting cosmic string: exact solution

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## Abstract

We solve the equations of motion and find the Lorentz transformation associated with a kink in superconducting cosmic string. The kink velocity does not depend on its amplitude. The kink amplitude cannot be arbitrary but it varies within definite range and determines the explicit form of the relevant Lorentz transformation.

## 1 Introduction

The behavior of current-carrying (or superconducting) cosmic string is determined by a pair of intrinsic equations of motion and a pair of extrinsic equations where intrinsic and extrinsic motions are considered separately [1]. These equations can develop three classes of perturbations [1, 2, 3, 4, 5, 6, 7], depending on the parameter which is subject to change: shocks (the current), kinks (the curvature), cusps (the spatial geometry). The shocks and kinks of infinitesimal amplitude were studied analytically [1]. Numerical simulations [4, 5] revealed the process of finite-amplitude shock and kink formation.

The numerical analysis becomes impossible when a stationary discontinuous solution is formed. The analytical analysis remains the only helpful tool, and it was already extended to the shocks with finite increment of the current  $\Delta\chi$  [8]. The kinks of finite amplitude (jump of the curvature  $\Delta\kappa^\mu$ ) is the subject of our present interest. We have recently adjusted the general

theory of discontinuities [9, 10] to the superconducting cosmic strings and derived the equations of intrinsic and extrinsic discontinuities [11]. Now, we are looking for explicit solution of these equations in application to the kinks.

## 2 Characteristic four-vector

The front of a discontinuity is a hypersurface, moving along the characteristic direction in (3+1) space-time [10]. The discontinuity is stable when its unit characteristic four-vector remains the same before (label "−") and behind (label "+") the front [11]:

$$\lambda_+^\mu = \lambda_-^\mu \equiv \alpha_+ \frac{w_+ u_+^\mu + v_+^\mu}{\sqrt{1 - w_+^2}} + \sigma_+ s_+^\mu = \alpha_- \frac{w_- u_-^\mu + v_-^\mu}{\sqrt{1 - w_-^2}} + \sigma_- s_-^\mu \quad (1)$$

$$\lambda_\pm^\mu \lambda_{\pm\mu} = 1 \quad \alpha_\pm^2 + \sigma_\pm^2 = \alpha_\mp^2 + \sigma_\mp^2 = 1 \quad (2)$$

where unit worldsheet four-vectors  $u_\pm^\mu$  ( $u_\pm^\mu u_{\pm\mu} = -1$ ),  $v_\pm^\mu$  ( $v_\pm^\mu v_{\pm\mu} = 1$ ) and normal four-vector  $s_\pm^\mu$  ( $s_\pm^\mu s_{\pm\mu} = 1$ ) are mutual orthogonal ( $u_\pm^\mu v_{\pm\mu} = u_\pm^\mu s_{\pm\mu} = s_\pm^\mu v_{\pm\mu} = 0$ ). The velocities before and behind the front ( $w_-$  and  $w_+$ ) are measured in the preferred reference frame, co-moving the discontinuity. When we operate in the laboratory reference frame, the front propagates at velocity  $W$  with respect to the string, and there is no motion before the front  $W_- = 0$  (however, the motion behind it  $W_+ \neq 0$  may occur). When we switch to the co-moving reference frame, associated with the front at rest, we operate with a finite flow before the front  $w_- = -W$ , and a flow behind the front is determined as  $w_+ = (W_+ - W) / (1 - Ww_+)$ .

There also exists a space-like unit four-vector  $b_\pm^\nu$ , which is orthogonal to the relevant three four-vectors in (1), namely:

$$b_\pm^\nu u_{\pm\nu} = b_\pm^\nu v_{\pm\nu} = b_\pm^\nu s_{\pm\nu} = 0 \quad b_\pm^\nu b_{\pm\nu} = 1 \quad (3)$$

so that the sets  $\{u_-^\nu, v_-^\nu, s_-^\nu, b_-^\nu\}$  and  $\{u_+^\nu, v_+^\nu, s_+^\nu, b_+^\nu\}$  constitute two orthonormal tetrads before and behind the discontinuity, and the relevant Gram matrices  $\mathfrak{J} = \text{diag}(-1, 1, 1, 1)$  coincide with the Minkowski tensor. The Lorentz transformations

$$\begin{pmatrix} u_+^\mu \\ v_+^\mu \\ s_+^\mu \\ b_+^\mu \end{pmatrix} = O \begin{pmatrix} u_-^\mu \\ v_-^\mu \\ s_-^\mu \\ b_-^\mu \end{pmatrix} \quad \begin{pmatrix} u_-^\mu \\ v_-^\mu \\ s_-^\mu \\ b_-^\mu \end{pmatrix} = O^{-1} \begin{pmatrix} u_+^\mu \\ v_+^\mu \\ s_+^\mu \\ b_+^\mu \end{pmatrix} \quad (4)$$

have unit determinant

$$\det O = \det O^{-1} = \pm 1 \quad (5)$$

and satisfy the orthogonality condition  $O^T \mathbf{J} O = \mathbf{J}$  that is equivalent to

$$OO^{-1} = E = 1 \quad O^{-1}O = \tilde{E} = 1 \quad (6)$$

where  $E = \tilde{E} = \text{diag}(1, 1, 1, 1)$ , and matrices

$$O = \begin{pmatrix} -Y & G & P & F \\ -H & X & Q & V \\ -A & R & S & I \\ -M & N & J & B \end{pmatrix} \quad O^{-1} = \begin{pmatrix} -Y & H & A & M \\ -G & X & R & N \\ -P & Q & S & J \\ -F & V & I & B \end{pmatrix} \quad (7)$$

include coefficients

$$G = v_-^\mu u_{+\mu} \quad H = u_-^\mu v_{+\mu} \quad X = v_-^\mu v_{+\mu} \quad Y = u_-^\mu u_{+\mu} \quad (8)$$

$$P = s_-^\mu u_{+\mu} \quad Q = s_-^\mu v_{+\mu} \quad A = s_+^\mu u_{-\mu} \quad R = s_+^\mu v_{-\mu} \quad (9)$$

$$F = b_-^\mu u_{+\mu} \quad V = b_-^\mu v_{+\mu} \quad M = b_+^\mu u_{-\mu} \quad N = b_+^\mu v_{-\mu} \quad (10)$$

$$I = b_-^\mu s_{+\mu} \quad J = b_+^\mu s_{-\mu} \quad S = s_-^\mu s_{+\mu} \quad B = b_+^\mu b_{-\mu} \quad (11)$$

Identical transformation is achieved by (4) in the infinitesimal limit

$$X \rightarrow -Y \rightarrow S \rightarrow B \rightarrow 1 \quad G \rightarrow 0 \quad (12)$$

$$P \rightarrow Q \rightarrow F \rightarrow V \rightarrow I \rightarrow 0 \quad (13)$$

$$A \rightarrow R \rightarrow M \rightarrow N \rightarrow J \rightarrow 0 \quad (14)$$

when  $w_+ \rightarrow w_-$  and

$$\alpha_+ \rightarrow \alpha_- \quad \sigma_+ \rightarrow \sigma_- \quad (15)$$

Multiplying equality (1) by each of four-vectors  $u_-^\mu$ ,  $v_-^\mu$ ,  $s_-^\mu$ ,  $b_-^\mu$ , we get a set of equations for the characteristic four-vector:

$$(Yw_+ + H) \frac{\alpha_+}{\sqrt{1-w_+^2}} + A\sigma_+ = -\frac{w_- \alpha_-}{\sqrt{1-w_-^2}} \quad (16)$$

$$(Gw_+ + X) \frac{\alpha_+}{\sqrt{1-w_+^2}} + R\sigma_+ = \frac{\alpha_-}{\sqrt{1-w_-^2}} \quad (17)$$

$$(Pw_+ + Q) \frac{\alpha_+}{\sqrt{1-w_+^2}} + S\sigma_+ = \sigma_- \quad (18)$$

$$(Fw_+ + V) \frac{\alpha_+}{\sqrt{1-w_+^2}} + I\sigma_+ = 0 \quad (19)$$

Multiplying equality (1) by four-vectors  $u_+^\nu$ ,  $v_+^\nu$ ,  $s_+^\nu$ ,  $b_+^\nu$ , we also get

$$(Yw_- + G) \frac{\alpha_-}{\sqrt{1-w_-^2}} + P\sigma_- = -\frac{w_+\alpha_+}{\sqrt{1-w_+^2}} \quad (20)$$

$$(Hw_- + X) \frac{\alpha_-}{\sqrt{1-w_-^2}} + Q\sigma_- = \frac{\alpha_+}{\sqrt{1-w_+^2}} \quad (21)$$

$$(Aw_- + R) \frac{\alpha_-}{\sqrt{1-w_-^2}} + S\sigma_- = \sigma_+ \quad (22)$$

$$(Mw_- + N) \frac{\alpha_-}{\sqrt{1-w_-^2}} + J\sigma_- = 0 \quad (23)$$

The curvature four-vector [12]

$$\kappa_\pm^\mu = \eta_{+\rho}^\nu \nabla_\nu \eta_+^{\mu\rho} \quad (24)$$

is determined by the projective tensor

$$\eta_+^{\mu\rho} = v_\pm^\mu v_\pm^\rho - u_\pm^\mu u_\pm^\rho \quad (25)$$

In the light of (4) and (7), the latter is presented in the form

$$\begin{aligned} \eta_+^{\mu\rho} = & (H^2 - Y^2) u_-^\mu u_-^\rho + (GY - XH) u_-^{(\mu} v_-^{\rho)} + (X^2 - G^2) v_-^\mu v_-^\rho + \\ & + (PY - QH) s_-^{(\mu} u_-^{\rho)} + (QX - PG) s_-^{(\mu} v_-^{\rho)} + (Q^2 - P^2) s_-^\mu s_-^\rho + \\ & + (FY - VH) b_-^{(\mu} u_-^{\rho)} + (VX - FG) b_-^{(\mu} v_-^{\rho)} + \\ & + (QV - PF) b_-^{(\mu} s_-^{\rho)} + (V^2 - F^2) b_-^\mu b_-^\rho \end{aligned} \quad (26)$$

A finite-amplitude kink is associated with a finite change of the curvature  $\kappa_+^\mu - \kappa_-^\mu \neq 0$ , and the projective tensor (25) and (26) cannot coincide at different sides of the discontinuity. As soon as we know the components of matrix (7), we know everything about the kink.

### 3 Kink solution of extrinsic equations

While the intrinsic equations of motion may admit discontinuities with alternating current (shocks), the extrinsic equations may admit discontinuity with alternating curvature (kinks) [4, 5]. Every discontinuous solution of the extrinsic equations of motion satisfies the following equations [11]:

$$\alpha_+ \{u_-^\mu - H v_+^\mu + Y u_+^\mu + w_+ (v_-^\mu - X v_+^\mu + G u_+^\mu)\} = 0 \quad (27)$$

$$\alpha_- \{u_+^\mu - G v_-^\mu + Y u_-^\mu + w_- (v_+^\mu - X v_-^\mu + H u_-^\mu)\} = 0 \quad (28)$$

$$\alpha_+ \{-w_+ (u_-^\mu - H v_+^\mu + Y u_+^\mu) - c_{E+}^2 (v_-^\mu - X v_+^\mu + G u_+^\mu)\} = 0 \quad (29)$$

$$\alpha_- \{-w_- (u_+^\mu - G v_-^\mu + Y u_-^\mu) - c_{E-}^2 (v_+^\mu - X v_-^\mu + H u_-^\mu)\} = 0 \quad (30)$$

where  $c_E \equiv c_{E-} = c_{E+}$  is the speed of infinitesimal extrinsic perturbations, and this speed is determined by the physical state of the string [1]. Since a kink solution admits the change of the curvature, while the current  $\chi_+ = \chi_-$  remains constant, this speed  $c_E$  is the same at both sides.

Equations (27)-(30) have evident degenerate solution

$$\alpha_+ = \alpha_- = 0 \quad (31)$$

The development of this degenerate discontinuity is not discussed now, it will be considered in a separate study devoted to the cusps. As for the non-degenerate kink discontinuities, functions  $\alpha_+$  or  $\alpha_-$  are not forbidden to vanish at some finite amplitude of the perturbation rather than in the infinitesimal limit (15).

If  $\alpha_+ \neq 0$  equations (27) and (29) have special solution

$$v_-^\mu = X v_+^\mu - G u_+^\mu \quad (32)$$

$$u_-^\mu = H v_+^\mu - Y u_+^\mu \quad (33)$$

that, in the light of transformations (4)-(7) gives

$$P = Q = F = V = 0 \quad (34)$$

$$A = R = M = N = 0 \quad (35)$$

$$Y = \pm X = \pm \sqrt{1 + G^2} \quad H = \pm G \quad (36)$$

The same solution (34)-(36) satisfies equations (28) and (30) if  $\alpha_- \neq 0$ . Substituting (34)-(36) in (26), we establish  $\eta_+^{\mu\rho} = \eta_-^{\mu\rho}$  that implies constant

curvature  $\kappa_+^\mu = \kappa_-^\mu$  (24) at any parameter  $G$ . Solution (34)-(36) cannot belong to a finite-amplitude kink.

If  $\alpha_+ \neq 0$  and solution (34)-(36) takes no place, equations (27) and (29) imply

$$w_+ = c_E \quad (37)$$

$$u_-^\mu - H v_+^\mu + Y u_+^\mu + c_E (v_-^\mu - X v_+^\mu + G u_+^\mu) = 0 \quad (38)$$

Multiplying equation (38) by  $u_-^\mu$ ,  $v_-^\mu$ ,  $s_-^\mu$ ,  $b_-^\mu$ ,  $u_+^\mu$ ,  $v_+^\mu$ ,  $s_+^\mu$ , and  $b_+^\mu$ , we obtain (omiting relations which are repeated):

$$-1 - (H + X c_E) H + (Y + G c_E) Y = 0 \quad (39)$$

$$c_E - (H + X c_E) X + (Y + G c_E) G = 0 \quad (40)$$

$$(H + X c_E) Q - (Y + G c_E) P = 0 \quad (41)$$

$$(H + X c_E) V - (Y + G c_E) F = 0 \quad (42)$$

$$A = -R c_E \quad (43)$$

$$M = -N c_E \quad (44)$$

If  $\alpha_- \neq 0$  and solution (34)-(36) takes no place, equations (28) and (30) imply

$$w_- = c_E \quad (45)$$

$$u_+^\mu - G v_-^\mu + Y u_-^\mu + c_E (v_+^\mu - X v_-^\mu + H u_-^\mu) = 0 \quad (46)$$

Multiplying equation (46) by  $u_-^\mu$ ,  $v_-^\mu$ ,  $s_-^\mu$ ,  $b_-^\mu$ ,  $u_+^\mu$ ,  $v_+^\mu$ ,  $s_+^\mu$ , and  $b_+^\mu$ , we obtain

$$-1 - (G + X c_E) G + (Y + H c_E) Y = 0 \quad (47)$$

$$c_E - (G + X c_E) X + (Y + H c_E) H = 0 \quad (48)$$

$$(G + X c_E) R - (Y + H c_E) A = 0 \quad (49)$$

$$(G + X c_E) N - (Y + H c_E) M = 0 \quad (50)$$

$$P = -Q c_E \quad (51)$$

$$F = -V c_E \quad (52)$$

Suppose that altogether  $\alpha_+ \neq 0$  and  $\alpha_- \neq 0$ . If  $Q \neq 0$ , equations (41) and (51) yield

$$H + X c_E = -c_E (Y + G c_E) \quad (53)$$

If  $Q = 0$ , equation (51) gives  $P = 0$ , and equations (20)-(21) yield the same constraint (53). Identity  $H + Xc_E = 0$  is impossible because it is not valid at the infinitesimal level  $X \rightarrow 1$  and  $H \rightarrow 0$ . Substituting (53) in equations (39)-(40), we obtain solution

$$Y = \pm 1 - Gc_E \quad X = \mp 1 - \frac{G}{c_E} \quad H = G \quad (54)$$

If  $\alpha_+ \neq 0$  and  $\alpha_- = 0$ , equations (20) and (21) result in (51). Substituting (51) in (41) we get (53) that together with (39)-(40) yields (54). Equation (23) implies  $J = 0$ , and taking components of identity (6), namely

$$E_{14} = (Yc_E + G)N + FB = 0 \quad (55)$$

$$E_{24} = (Gc_E + X)N + VB = 0 \quad (56)$$

together with (54), we have

$$(F + Vc_E)B = 0 \quad (57)$$

Since  $B = 0$  contradicts to infinitesimal limit  $B \rightarrow 1$ , we must put  $B \neq 0$  and obtain equality (52).

If  $\alpha_- \neq 0$  and  $\alpha_+ = 0$ , equations (16) and (17) result in (43). Substituting (43) in (49) we get

$$G + Xc_E = -c_E(Y + Hc_E) \quad (58)$$

where identity  $G + Xc_E = 0$  is impossible. Substituting (58) in equations (47)-(48), we obtain the same solution (54). Equation (19) implies  $I = 0$ , making use of relations incorporated in identity (6), namely

$$\tilde{E}_{14} = (Yc_E + G)N + MB = 0 \quad (59)$$

$$\tilde{E}_{24} = (Gc_E + X)N + NB = 0 \quad (60)$$

together with (54), we have

$$(M + Nc_E)B = 0 \quad (61)$$

Since  $B = 0$  contradicts to infinitesimal limit  $B \rightarrow 1$ , we must put  $B \neq 0$  and obtain equality (44).

Thus, solution (54) takes place together with (43)-(44), (51)-(52). Solution (54) at the lower sign

$$Y = -1 - Gc_E \quad X = 1 - \frac{G}{c_E} \quad H = G \quad (62)$$

corresponds to identity  $u_+^\mu = u_-^\mu$  and  $v_+^\mu = v_-^\mu$  in the infinitesimal limit (12)-(14). Solution (54) with the upper sign corresponds to  $u_+^\mu = -u_-^\mu$  and  $v_+^\mu = -v_-^\mu$  and implies space-time reversal, which was not discovered in the infinitesimal perturbations [1]. Therefore, this solution is not realized, and we have to deal with the only one (62).

By means of relations (43)-(44), (51)-(52) we present matrices (7) in the form

$$O = \begin{pmatrix} -Y & G & -Qc_E & -Vc_E \\ -G & X & Q & V \\ Rc_E & R & S & I \\ Nc_E & N & J & B \end{pmatrix} \quad O^{-1} = \begin{pmatrix} -Y & G & -Rc_E & -Nc_E \\ -G & X & R & N \\ Qc_E & Q & S & J \\ Vc_E & V & I & B \end{pmatrix} \quad (63)$$

Substituting (63) in (6) we have

$$E_{11} = Y^2 - G^2 - (Q^2 + V^2) c_E^2 = 1 \quad (64)$$

$$E_{22} = X^2 - G^2 + Q^2 + V^2 = 1 \quad (65)$$

so that  $Y^2 \geq 1$  and  $Y^2 \geq X^2$ . In the light of (62), these inequalities are satisfied when

$$0 \leq G \leq G_{\max} = \frac{2c_E}{1 - c_E^2} \quad -1 \leq g \leq 1 \quad (66)$$

where we define

$$g = Gc_E + X = 1 + Gc_E - \frac{G}{c_E} \quad (67)$$

Since  $Y < -1$  at any non-negative  $G$ , the Lorentz transformation (4) is always orthochronous. Median value

$$G_0 = \frac{c_E}{1 - c_E^2} \quad X_0 = -\frac{c_E^2}{1 - c_E^2} < 0 \quad (68)$$

is equivalent to  $g = 0$ . When the current tends to the chiral limit  $\chi \rightarrow 0$ , the velocity of extrinsic perturbations tends to the speed of light  $c_E \rightarrow 1$  [1, 4, 5] that corresponds to  $G_0 \rightarrow \infty$  and  $G_{\max} \rightarrow \infty$ .



## 4 Parameters of kink

Substituting (62) and (63) in (6) we obtain a system of algebraic equations. Omitting the evident repeated expressions, we write down the following equations

$$E_{12} = G(g+1) - c_E(Q^2 + V^2) \quad (69)$$

$$E_{23} = gR + QS + VI = 0 \quad (70)$$

$$E_{24} = gN + QJ + VB = 0 \quad (71)$$

$$E_{33} = (1 - c_E^2)R^2 + S^2 + I^2 = 1 \quad (72)$$

$$E_{34} = (1 - c_E^2)RN + SJ + IB = 0 \quad (73)$$

$$E_{44} = (1 - c_E^2)N^2 + J^2 + B^2 = 1 \quad (74)$$

$$\tilde{E}_{12} = G(g+1) - c_E(R^2 + N^2) = 0 \quad (75)$$

$$\tilde{E}_{23} = gQ + RS + NJ = 0 \quad (76)$$

$$\tilde{E}_{24} = gV + RI + NB = 0 \quad (77)$$

$$\tilde{E}_{33} = (1 - c_E^2)Q^2 + S^2 + J^2 = 1 \quad (78)$$

$$\tilde{E}_{34} = (1 - c_E^2)QV + SI + JB = 0 \quad (79)$$

$$\tilde{E}_{44} = (1 - c_E^2)V^2 + I^2 + B^2 = 1 \quad (80)$$

where  $g$  is defined in (67).

The unitary determinant (5) of the Lorentz matrices (63) is:

$$(G^2 - YX)(BS - IJ) + \{(Y + c_E G) + c_E(c_E X + G)\} \{R(QB - VJ) + N(VS - QI)\} = \pm 1 \quad (81)$$

Substituting (62), (67) and (69), (70), (71) in (81), we have

$$BS - IJ = \pm g \quad (82)$$

that is equivalent to

$$(S^2 + J^2)(I^2 + B^2) - (IS + JB)^2 = g^2 \quad (83)$$

and contains no additional information because it is a consequence of equations (69), (78), (79), (80).

Taking into account solution (43)-(44), (51)-(52) and (62), we simplify equations (16)-(23) in the following form

$$g\alpha_+ + \sqrt{1 - c_E^2} R\sigma_+ = \alpha_- \quad (84)$$

$$\sqrt{1 - c_E^2} Q\alpha_+ + S\sigma_+ = \sigma_- \quad (85)$$

$$\sqrt{1 - c_E^2} V\alpha_+ + I\sigma_+ = 0 \quad (86)$$

$$g\alpha_- + \sqrt{1 - c_E^2} Q\sigma_- = \alpha_+ \quad (87)$$

$$\sqrt{1 - c_E^2} R\alpha_- + S\sigma_- = \sigma_+ \quad (88)$$

$$\sqrt{1 - c_E^2} N\alpha_- + J\sigma_- = 0 \quad (89)$$

Equations (84)-(89) determine the unknowns  $\alpha_{\pm}$  and  $\sigma_{\pm}$  but they do not provide enough information to equations (69)-(80).

It is easy to check that equation (73) is a consequence of (70), (72), (77), (79), (80). In turn, equation (77) is a consequence of (69), (70), (71), (78), (80). Equation (74) follows from (69), (72), (75), (78), (80). But equation (75) follows from (70), (71), (78), (79), (80). Equation (76) is derived from (69), (70), (71), (78), (79). After all, taking equations (69), (70), (78), (79), (80), we come to equation (79). As a result, there are only 6 independent equations (69), (70), (71), (72), (78), and (80), while there are 9 unknowns  $Q, R, N, V, S, B, I, J, G$ . These 6 independent equations allow to determine 6 variables, while the rest 3 variables, for example,  $Q, R, G$  cannot be reduced or expressed through each other. As soon as we know the triplet  $\{Q, R, G\}$ , all other unknowns are immediately established.

Nevertheless, the system must be fully resolvable because we have started with two extrinsic equations of motion and two independent parameters. The first parameter is the kink velocity  $c_E$ , it is already determined by the physical state of the string. The second parameter is an increment of the curvature  $\Delta\kappa^\mu$ , it can be reflected by a single variable. Therefore, unknowns  $\{Q, R, G\}$  are no more than functions, depending on this singular argument. The most natural choice concerns  $G$ , and we must establish functions  $Q(G)$  and  $R(G)$ .

## 5 Energy of kink

Defining

$$Q = h \sin a \quad V = h \cos a \quad (90)$$

$$R = h \sin x \quad N = h \cos x \quad (91)$$

where

$$h^2 = \frac{G(g+1)}{c_E} = \frac{1-g^2}{1-c_E^2} = \frac{(1-c_E^2)}{c_E^2} G(G_{\max} - G) \quad (92)$$

and  $G_{\max}$  is taken from (66), we present equations (69) and (75) in the form

$$Q^2 + V^2 = R^2 + N^2 = h^2 \quad (93)$$

Function  $h(G)$  disappears when  $G = 0$  ( $g = 1$ ) or when  $G = G_{\max}$  ( $g = -1$ ) that takes place when and only when

$$Q = V = 0 \quad \Leftrightarrow \quad R = N = 0 \quad (94)$$

In the light of (43)-(44) and (51)-(52), it corresponds to (34)-(35), while equalities (64)-(65) imply  $Y^2 = X^2 = 1 + G^2$ . As a result, the projective tensor (26) remains unchanged and the curvature is constant when  $h = 0$ .

A triplet of unknowns  $\{a, x, h\}$  is equivalent to  $\{Q, R, G\}$ . Defining four-vectors

$$h_-^\mu = (Qs_-^\mu + Vb_-^\mu) = h(s_-^\mu \sin a + b_-^\mu \cos a) \quad (95)$$

$$h_+^\mu = (Rs_+^\mu + Nb_+^\mu) = h(s_+^\mu \sin x + b_+^\mu \cos x) \quad (96)$$

$$\zeta_\pm^\mu = G(c_E u_\pm^\mu + v_\pm^\mu) - c_E h_\pm^\mu \quad (97)$$

where

$$\zeta_\pm^\mu \zeta_{\pm\mu} = 2Gw \quad h_\pm^\mu h_{\pm\mu} = h^2 \quad (98)$$

we substitute them in (4), (63) and obtain relations

$$\zeta_+^\mu = -\zeta_-^\mu \quad (99)$$

$$u_\pm^\mu = u_\mp^\mu + \zeta_\mp^\mu \quad v_\pm^\mu = v_\mp^\mu - \frac{\zeta_\pm^\mu}{c_E} \quad (100)$$

Substituting (100) in (25), we determine the projective tensor behind the discontinuity

$$\eta_+^{\mu\rho} = \left(v_-^\mu - \frac{\zeta_-^\mu}{c_E}\right) \left(v_-^\rho - \frac{\zeta_-^\rho}{c_E}\right) - (u_-^\mu + \zeta_-^\mu)(u_-^\rho + \zeta_-^\rho) \quad (101)$$

that is

$$\eta_+^{\mu\rho} = \eta_-^{\mu\rho} - \frac{v_-^{(\mu}\zeta_-^{\rho)}}{c_E} - u_-^{(\mu}\zeta_-^{\rho)} + \frac{1 - c_E^2}{c_E^2}\zeta_-^\mu\zeta_-^\rho \quad (102)$$

In the light of (4) and (63), we established an invariant time-like bicharacteristic four-vector

$$\xi^\mu = \xi_-^\mu \equiv u_-^\mu + c_E v_-^\mu = \xi_+^\mu \equiv u_+^\mu + c_E v_+^\mu \quad (103)$$

which is orthogonal to the characteristic four-vector (1) and to four-vectors (95)-(97).

The stress-energy tensor of a cosmic string [12]

$$\bar{T}_\pm^{\mu\nu} = U u_\pm^\mu u_\pm^\nu - T v_\pm^\mu v_\pm^\nu = U (u_\pm^\mu u_\pm^\nu - c_E^2 v_\pm^\mu v_\pm^\nu) \quad (104)$$

includes the same values  $U_+ = U_- \equiv U$  and  $T_+ = T_- \equiv T$  because the current  $\chi$  is the same at both sides of the discontinuity. Taking into account expressions (100) and (103), we have

$$\bar{T}_+^{\mu\nu} = \bar{T}_-^{\mu\nu} + U \xi_-^{(\mu}\zeta_-^{\nu)} \quad (105)$$

The energy density

$$E_+ = \bar{T}_+^{00} = E_- + 2U\xi_-^0\zeta_-^0 = E_- - 2U\xi_+^0\zeta_+^0 \quad (106)$$

in the light of (95)-(97), is written so:

$$E_+ = E_- + E_0 (w u_-^0 + v_-^0) G - E_0 (s_-^0 \sin a + b_-^0 \cos a) c_E h(G) \quad (107)$$

$$E_+ = E_- - E_0 (w u_+^0 + v_+^0) G + E_0 (s_+^0 \sin x + b_+^0 \cos x) c_E h(G) \quad (108)$$

where

$$E_0 = 2U\xi_-^0 = 2U\xi_+^0 \quad (109)$$

is independent on the kink parameters because it is determined by the string worldsheet and the current which circulates within it.

We consider a finite-amplitude stationary kink with constant characteristic four-vector (1) rather than a kink in the process of formation and development. As a matter of fact, as soon as a kink is formed, the energy is not conserved [4, 5]. Although this stationary kink is responsible for transition between two equilibrium states of the string, the energy is not conserved ( $E_+ \neq E_-$ ) because it is not a continuous solution of the equations of motion.

The same variational principle cannot be applied immediately at both sides of the discontinuity, but the energy extremum takes place with respect to the relevant floating parameters at each side. It implies the following.

The energy functional (107) becomes a unique function  $E_+ \{G\}$  when the floating variable  $a$  is adjusted at the extremum level

$$\frac{\partial E_+}{\partial a} = 0 \quad \Rightarrow \quad s_-^0 \cos a = b_-^0 \sin a \quad \Leftrightarrow \quad s_-^0 V = b_-^0 Q \quad (110)$$

In the view of (4) and (63), equality (110) implies

$$(NS - RJ) s_-^0 = 0 \quad (NI - RB) b_-^0 = 0 \quad (111)$$

There is no principal difference between the sides of the discontinuity: in contrast to the shock waves [8], the evolutionary condition imposes no restriction on the discontinuity transition in both directions (from "++" to "--" and the inverse direction). However, we cannot deal with fixed  $u_-^\mu$ ,  $v_-^\mu$ ,  $s_-^\mu$ ,  $b_-^\mu$  and fixed  $u_+^\mu$ ,  $v_+^\mu$ ,  $s_+^\mu$ ,  $b_+^\mu$  simultaneously, since only one set is given, while the other set is determined by the kink, according to the Lorentz transformation (4). As soon as we operate with given four-vectors  $u_+^\mu$ ,  $v_+^\mu$ ,  $s_+^\mu$ ,  $b_+^\mu$  as a primary tetrad, the energy functional (108) is considered as unique function  $E_+ \{G\}$  where the floating variable  $x$  is adjusted at the extremum level

$$\frac{\partial E_+}{\partial x} = 0 \quad \Rightarrow \quad s_+^0 \cos x = b_+^0 \sin x \quad \Leftrightarrow \quad s_+^0 N = b_+^0 R \quad (112)$$

In the view of (4) and (63), equality (112) implies

$$(VS - QJ) s_+^0 = 0 \quad (VI - QB) b_+^0 = 0 \quad (113)$$

By means of (4) and (63), we establish the following possibilities

$$s_{-0} = 0 \quad \vec{\xi}_- \cdot \vec{s}_- = w (\vec{u}_+^2 + \vec{v}_+^2) Q = 0 \quad \Rightarrow \quad Q = 0 \quad (114)$$

$$b_{-0} = 0 \quad \vec{\xi}_- \cdot \vec{b}_- = w (\vec{u}_+^2 + \vec{v}_+^2) V = 0 \quad \Rightarrow \quad V = 0 \quad (115)$$

$$s_{+0} = 0 \quad \vec{\xi}_+ \cdot \vec{s}_+ = w (\vec{u}_+^2 + \vec{v}_+^2) R = 0 \quad \Rightarrow \quad R = 0 \quad (116)$$

$$b_{+0} = 0 \quad \vec{\xi}_+ \cdot \vec{b}_+ = w (\vec{u}_+^2 + \vec{v}_+^2) N = 0 \quad \Rightarrow \quad N = 0 \quad (117)$$

where vectors  $\vec{u}_\pm$ ,  $\vec{v}_\pm$ ,  $\vec{s}_\pm$ ,  $\vec{b}_\pm$  and  $\vec{\xi}_\pm = \vec{u}_\pm + c_E \vec{v}_\pm$  are spatial components of the relevant four-vectors. In the view of (94) and (114)-(117), it is clear

that simultaneous  $s_-^0 = b_-^0 = 0$  or simultaneous  $s_+^0 = b_+^0 = 0$  corresponds to constant curvature at zero kink amplitude  $G = 0$ . Equations (111) and (113) are satisfied without regard of  $s_\pm^0$  and  $b_\pm^0$  when, again, identity (94) takes place and the curvature remains constant. After all, simultaneous non-zero  $s_-^0 \neq 0$ ,  $b_-^0 \neq 0$  in (111) or simultaneous non-zero  $s_+^0 \neq 0$ ,  $b_+^0 \neq 0$  in (113) implies  $BS - IJ = 0$ , in the light of (82), it corresponds to particular  $g = 0$  that does not embrace the infinitesimal limit at  $g \rightarrow 1$  ( $G \rightarrow 0$ ).

Therefore, a kink of finite amplitude can satisfy equations (111) and (113) under one of the following conditions:

$$s_{-0} = s_{+0} = 0 \quad Q = R = 0 \quad (118)$$

$$s_{-0} = b_{+0} = 0 \quad Q = N = 0 \quad (119)$$

$$b_{-0} = s_{+0} = 0 \quad V = R = 0 \quad (120)$$

$$b_{-0} = b_{+0} = 0 \quad V = N = 0 \quad (121)$$

As a matter of fact, when 8 components of four-vectors  $s_-^\mu$  and  $b_-^\mu$  are subject to 7 constraints, and 1 component remains uncertain, the discontinuity is running in uncertain direction. The problem remains disambiguous until we impose additional self-consistent constraint associated with extremum of the energy density (107) where only one component ( $s_-^0$  or  $b_-^0$ ) is free to float, while the other component is fixed at a value which provides the energy minimum. The same view concerns the components of four-vectors  $s_+^\mu$  and  $b_+^\mu$ .

## 6 Explicit kink solution

Substituting constraint (118) in equations (84) and (87), we find  $g^2 = 1$  (because we do not consider the degenerate case  $\alpha_- = 0$ ) that, in the light of (92), corresponds to constant curvature. Substituting (119) in equation (71), we find  $VB = 0$  where solution  $B = 0$  contradicts to the infinitesimal limit (12) while solution  $V = 0$ , in the view of (69) and (119), implies constant curvature. Substituting (120) in equation (77), we find  $NB = 0$  where solution  $B = 0$  is impossible while solution  $V = 0$ , in the view of (75) and (120), again implies constant curvature.

Substituting constraint (121) in equations (69) and (75) we find

$$R^2 = Q^2 = \frac{1 - g^2}{1 - c_E^2} \quad (122)$$

where  $Q = R = 0$  at constant curvature. Substituting constraint (121) in equations (71) and (77), we find

$$I = J = 0 \quad (123)$$

Substituting (11) in equation (70) and (74), we find  $S = \pm g$  and  $B^2 = 1$  that, in the view of infinitesimal limit (12) at  $g \rightarrow 1$  (67), implies

$$S = g \quad B = 1 \quad (124)$$

Substituting (124) back in (70), we find

$$R = -Q = \pm \sqrt{G \left( 2 - G \frac{(1 - c_E^2)}{c_E^2} \right)} \quad (125)$$

Substituting solution (122)-(125) in equations (84)-(89), we have two independent equations

$$\alpha_+ = g\alpha_- + \sqrt{1 - w^2}Q\sigma_- \quad (126)$$

$$\sigma_+ = -\sqrt{1 - w^2}Q\alpha_- + g\sigma_- \quad (127)$$

Solution (122)-(127) describes a kink of finite amplitude.

Note that conditions of extremum (110) and (112) allow to present the energy density (107)-(108) in the form of functional

$$E_+[G] = E_- + E_0 (c_E u_-^0 + v_-^0) G - E_0 c_E s_-^0 Q \quad (128)$$

where  $Q()$  is a unique function on  $G$ . As we have mentioned above, a kink can propagate in both directions that is reflected in the choice of the sign in formula (125). The energy density (107)-(108) of low-amplitude kinks at small  $G$  and  $s_+^0 \simeq s_-^0 = s^0$  is given by expression

$$E_+ \simeq E_- - E_0 c_E s^0 Q \simeq E_- \mp E_0 s^0 \sqrt{2G c_E} \quad (129)$$

Inequality  $E_+ < E_-$  implies an energetically favorable regime of propagation from "−" to "+". Opposite inequality  $E_+ > E_-$  corresponds to the kink propagation from "+" to "−". The critical kink development at  $G = G_{\max}$  in (66) takes place at  $g = S = -1$  in (124) and corresponds to constant curvature. The maximum absolute amplitude

$$R = -Q = \pm \frac{1}{\sqrt{1 - c_E^2}} \quad (130)$$

is achieved at the median value  $G = G_0$  (68), corresponding to  $g = S = 0$ . Indeed, function  $R(g) = -Q(g)$  from (122) is symmetric around its maximum at  $g = 0$  and comes to minimum at  $g = \pm 1$ .

## 7 Conclusion

In contrast to the shocks, the kinks require much more complicated analysis. The equations of extrinsic discontinuities (27)-(30) give no evident hint to explicit solution. We have to consider the Lorentz transformation (4) between the orthonormal tetrads before and behind the discontinuity, where the matrices of transformation (7) are not arbitrary but depend on the kink amplitude. The latter is associated with parameter  $G$ , which is not arbitrary but can vary within a definite finite range (66). According to the energy extremum, we establish the elements of the transformation (63) (62), (122)-(125). The kink velocity coincides with the speed of infinitesimal extrinsic perturbations  $c_E$  (37), (45) and does not depend on the kink amplitude  $G$ . The spatial geometry of the string is changing within a 2-dimensional plane because four-vector  $b_{\pm}^{\mu}$  remains constant at both sides of the discontinuity.

Having derived the kink solution (4), (63) (62), (122)-(125), we are ready to look for its further application by means of equations (24)-(26), (126)-(127). Now, the following problem deserves consideration: an explicit link between the kink parameter  $G$ , the relative increment of the curvature  $\Delta\kappa^{\mu}/\kappa_{-}^{\mu}$  and the angle in the kink vertex.

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